Control Systems II

System Modelling

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \frac{\partial f_2}{\partial u} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}$$

Stability

Lyapunov asymptotic stability $\Longleftrightarrow$ BIBO stable + Contollable Observable

State-Space to Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = C(A - s\mathbb{I})^{-1}B + D = \frac{C \cdot \operatorname{Adj}(A - s\mathbb{I}) \cdot B}{\det(A - s\mathbb{I})} + D = \frac{b_m s^m + \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}$$

Anfangswert: $\lim_{t\to 0^+} x(t) = \lim_{s\to\infty} s \cdot X(s)$

Endwert: $\lim_{t\to\infty} x(t) = \lim_{s\to 0} s \cdot X(s)$

Reachable Canonical Form $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

$m<n \rightarrow \left[\begin{array}{cccccc|c} 0 & 1 & 0 & \cdots & & 0 & 0 \\ \vdots & & \ddots & & & 1 & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} & 1 \\ \hline b_0 & b_1 & \cdots & b_m & & 0 & 0 \end{array}\right]$

$m=n \rightarrow \left[\begin{array}{ccc|c} 0 & 1 \ddots & 0 & 0 \\ & & 1 & \vdots \\ -a_0 & & -a_{n-1} & 1 \\ \hline b_0 - b_n a_0 & \cdots & b_{n-1} - b_n a_{n-1} & b_n \end{array}\right]$

Reachability / Observability

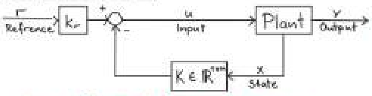
$R = [B \ AB \cdots A^{n-1}B]$ $\quad O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$

- $\det(R) \neq 0 \neq \det(O)$

Important Math

- $\angle\left(\frac{N}{D}\right) = \angle(N) - \angle(D)$, $\angle(a+b)^n = n\angle(a+b)$
- $\angle(z) = \tan^{-1}\left(\frac{Im}{Re}\right)$, $\operatorname{Adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

State Feedback



$$\dot{x}(t) = \underbrace{(A - BK)}_{A_{cc}}x(t) + Bk_r r(t)$$

Refrence Tracking

$$k_r = \left[-C(A-BK)^{-1}B\right]^{-1} = \frac{a_0 + k_0}{b_0}$$

Achermann Formula (Regulator)

$$K = [0\,0 \dots 1]R^{-1}p_{cc}^*(A) = [k_{n-1} \cdots k_0]$$

Poles placement (2.Order System)

$P(s) = \frac{k\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$, $\lambda_{1/2} = -\zeta\omega_0 \pm \omega_0\sqrt{\zeta^2 - 1}$

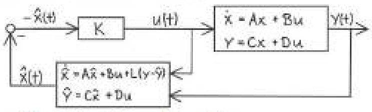
ζ	0.5	$1/\sqrt{2}$	1
T_r (rise time)	$1.8/\omega_0$	$2.2/\omega_0$	$2.7/\omega_0$
M_p (Overshoot)	16%	4%	0%
T_s (settling time)	$8.0/\omega_0$	$5.6/\omega_0$	$4.0/\omega_0$

$$p_{cc}^*(s) \stackrel{!}{=} p_{cc}(s)$$

Higher Order Pole Placement

1. Place dominant pair as usual.
2. Other poles 4x faster than dominant.

State Estimation



- $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$
  $= A\hat{x} + Bu + LC(x - \hat{x})$
  $= (A - LC)\hat{x} + Bu - Ly$

$\hat{e}(t) = x(t) - \hat{x}(t)$ | $\dot{\hat{e}}(t) = (A - LC)\hat{e}(t)$

Achermann Formula (Observer)

$$L = p_{cc}^*(A)O^{-1}[0\,0 \cdots 1]^T$$

Design Guidelines for Observer

- Make Observer 10x faster than Contoller
- ↑ Oberver gain ~ ↑ Measurement noise
- fast Observer ~ large error in transient

Separation Principle

$$\tilde{x} = \begin{bmatrix} x \\ e \end{bmatrix} \quad \dot{\tilde{x}} = \begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A-BK & BK \\ 0 & A-LC \end{bmatrix}}_{A_{cc}}\tilde{x}$$

Control & Estimation do not interact and can be designed independently.

$$(A - BK) \cdot 10 = (A - LC)$$

Popov-Belevitch-Hautus (PBH) Test

Controllability / Stabilizability:

- $\operatorname{rank}[\lambda\mathbb{I} - A \quad B] = n$, $\forall\lambda \in \mathbb{C}$
- $\operatorname{rank}[\lambda\mathbb{I} - A \quad B] = n$, $\forall\lambda \in \mathbb{C}^+$

Obserability / Detectability:

- $\operatorname{rank}\begin{bmatrix} \lambda\mathbb{I} - A \\ C \end{bmatrix} = n$, $\forall\lambda \in \mathbb{C}$
- $\operatorname{rank}\begin{bmatrix} \lambda\mathbb{I} - A \\ C \end{bmatrix} = n$, $\forall\lambda \in \mathbb{C}^+$

Reachability Decomposition

$\operatorname{rank}(R) = r < n$, $\quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx$

- reachable: $z_1 \in \mathbb{R}^r$ $\quad$ • not: $z_2 \in \mathbb{R}^{n-r}$
- $T^{-1} = \left[\underbrace{v_1, \dots, v_r}_{\text{Basis of } R} \,\middle|\, \underbrace{v_{r+1}, \dots, v_n}_{\text{completion}}\right]$

→ $v_1, \dots v_r$: linear independent Colums of R

$$\dot{z}_1 = \tilde{A}_{11}z_1 + \tilde{A}_{12}z_2 + \tilde{B}_1 u$$
$$\dot{z}_2 = \tilde{A}_{21}z_2$$
$$y = \tilde{C}_1 z_1 + \tilde{C}_2 z_2 + Du$$

- System $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, D)$ is reachable

Observability Decomposition

$\operatorname{rank}(O) = r < n$, $\quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = Tx$

- observable: $z_1 \in \mathbb{R}^r$ $\quad$ • not: $z_2 \in \mathbb{R}^{n-r}$
- $T^{-1} = \left[\underbrace{v_1, \dots, v_r}_{\text{completion}} \,\middle|\, \underbrace{v_{r+1}, \dots, v_n}_{\text{Basis of } \ker(O)}\right]$, $\quad T = \begin{bmatrix} u_1 \\ u_r \\ \text{completion} \end{bmatrix}$

→ $u_1, \dots u_r$: linear independent Rows of O

$$\dot{z}_1 = \tilde{A}_{11}z_1 + \tilde{B}_1 u$$
$$\dot{z}_2 = \tilde{A}_{21}z_2 + \tilde{A}_{22}z_2 + \tilde{B}_2 u$$
$$y = \tilde{C}_1 z_1 + Du$$

- System $(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, D)$ is observable

Kalman Decomposition

- System (A, B, C) $\quad z = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = Tx$

① $Im(R) = \{R_1, R_2\}$ & $\ker(O) = \{O_1, O_2\}$

② $H = [Im(R) \,|\, \ker(O)] = [R_1\, R_2\, O_1\, O_2]$

③ $\ker(H) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

④ Calculate $X_{r\bar{o}}, X_{ro}, X_{\bar{r}\bar{o}}, X_{\bar{r}o}$:

- $X_{r\bar{o}} = \operatorname{span}\{aR_1 + bR_2\} = \operatorname{span}\{cO_1 + dO_2\}$
- X_{ro}: $X_{r\bar{o}} \oplus X_{ro} = Im(R)$
- $X_{\bar{r}\bar{o}}$: $X_{r\bar{o}} \oplus X_{\bar{r}\bar{o}} = \ker(O)$
- $X_{\bar{r}o}$: $X_{r\bar{o}} \oplus X_{ro} \oplus X_{\bar{r}\bar{o}} \oplus X_{\bar{r}o} = \mathbb{R}^n$

⑤ $T^{-1} = [X_{r\bar{o}} \,|\, X_{ro} \,|\, X_{\bar{r}\bar{o}} \,|\, X_{\bar{r}o}]$, $T = \dots$

⑥ $\tilde{A} = TAT^{-1}$, $\tilde{B} = TB$, $\tilde{C} = CT^{-1}$

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{r\bar{o}} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ & \tilde{A}_{ro} & & \tilde{A}_{24} \\ & & \tilde{A}_{\bar{r}\bar{o}} & \tilde{A}_{34} \\ & & & \tilde{A}_{\bar{r}o} \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \\ \\ \end{bmatrix}$$

$$\tilde{C} = [\quad \tilde{C}_2 \quad \tilde{C}_4 \quad]$$

continued →

$$\tilde{A}_{reach} = \begin{bmatrix} \tilde{A}_{r\bar{o}} & \tilde{A}_{12} \\ 0 & \tilde{A}_{ro} \end{bmatrix}, \quad \tilde{A}_{obs} = \begin{bmatrix} \tilde{A}_{ro} & \tilde{A}_{24} \\ 0 & \tilde{A}_{\bar{r}o} \end{bmatrix}$$

$$\tilde{B}_{reach} = [\tilde{B}_1 \ \tilde{B}_2]^T, \quad C_{obs} = [\tilde{C}_2 \ \tilde{C}_4]$$

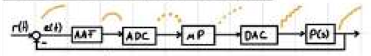
⑦ $p_{reach}(s) \stackrel{!}{=} \det[(\tilde{A}_{reach} - \tilde{B}_{reach}\tilde{K}_{reach}) - s\mathbb{I}]$

$\Longrightarrow K = \tilde{K}T$

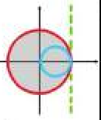
⑧ $p_{obs}(s) \stackrel{!}{=} \det[(\tilde{A}_{obs} - \tilde{L}_{obs}\tilde{C}_{obs}) - s\mathbb{I}]$

$\Longrightarrow L = T^{-1}\tilde{L}$

Discrete-time Control



- Exakt: $s = \frac{1}{T_s} \cdot \ln(z)$, $z = e^{s \cdot T_s}$
- Euler f: $s = \frac{z-1}{T_s}$, $z = s \cdot T_s + 1$
- Euler b: $s = \frac{z-1}{z \cdot T_s}$, $z = \frac{1}{1 - sT_s}$
- Tustin: $s = \frac{2}{T_s} \cdot \frac{z-1}{z+1}$, $z = \frac{1 + s \cdot \frac{T_s}{2}}{1 - s \cdot \frac{T_s}{2}}$



Sampling Theorem

- No Aliasing: $T_s < \frac{1}{2f_0}$, $f = \frac{\omega}{2\pi}$
- Theory: $f_s > 2f_{max}$, $T_s < \frac{T^*}{2}$
- Practice: $f_s > 10f_{max}$, $T_s < \frac{T^*}{10}$

Z-Transform

- $Z\{x[k]\} = X(z) = \sum_{-\infty}^{\infty} x[k] \cdot z^{-k}$
- $Z\{a_1x_1[k] + a_2x_2[k]\} = a_1X_1(z) + a_2X_2(z)$
- $Z\{x[k-b]\} = X(z)z^{-b}$
- $zX(z) = Z\{x[k+1]\}$

MIMO State-Space

$$\begin{bmatrix} \dot{x} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} B \end{bmatrix} \cdot \begin{bmatrix} u \end{bmatrix}$$

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} C \end{bmatrix} \cdot \begin{bmatrix} x \end{bmatrix} + \begin{bmatrix} D \end{bmatrix} \cdot \begin{bmatrix} u \end{bmatrix}$$

MIMO Transfer-Function

$$\begin{bmatrix} y_1 \\ \vdots \\ y_l \end{bmatrix} = \begin{bmatrix} P_{1,1}(s) & \cdots & P_{1,m}(s) \\ \vdots & & \vdots \\ P_{l,1}(s) & \cdots & P_{l,m}(s) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$P(s) = C(A - s\mathbb{I})^{-1}B$$

MIMO Poles/Zeros

Ex: $P(s) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$

1.order minors: a, b, c, d, e, f

2.order minors: $\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\det\begin{bmatrix} b & c \\ e & f \end{bmatrix}$, $\det\begin{bmatrix} a & c \\ d & f \end{bmatrix}$

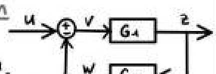
Poles: least common denom. of all minors!

Zeros: greatest common factor of ×!

↳ max order minor $= \frac{\times}{}$

- A MIMO-System can have a Pole & a Zero at the same Frequency without a Pole-Zero cancellation?
- Only possible if there also in the same direction!

MIMO-Interconnection



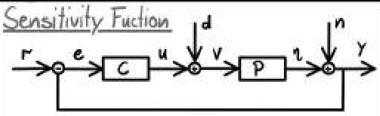
Input

$z = (\mathbb{I} \mp G_1G_2)^{-1}G_1 u$

Output

$v = (\mathbb{I} \mp G_2G_1)^{-1}u$

Sensitivity Fuction



Loop Transfer: $L_0 = PC$, $L_1 = CP$

Sensitivity-Output $(n \to y)$: $S_0 = (\mathbb{I} + L_0)^{-1}$

Sensitivity-Input $(d \to v)$: $S_1 = (\mathbb{I} + L_1)^{-1}$

Complementary-Output $(r \to y)$: $T_0 = (\mathbb{I} + L_0)^{-1}L_0$

Complementary-Input $(d \to -u)$: $T_1 = (\mathbb{I} + L_1)^{-1}L_1$

Norms

- $\|x\|_p = \left(\sum_i |x_i|^p\right)^{\frac{1}{p}}$ $\quad$ • $\|x\|_\infty = \max_i |x_i|$
- $\|A\|_\infty = \max_{ij} |a_{ij}|$
- $\|A\|_F = \sqrt{\sum_{ij} |a_{ij}|^2} = \sqrt{tr(G^*G)}$ ↳ Sum of diag. Elements

Induced Norm:

- $\|A\|_{i,2} = \sqrt{|\lambda_{max}(A^*A)|} = \bar{\sigma}(A)$

Signal Norms: $e(t) = [e_1(t), \dots, e_n(t)]^T$

- $\|e(t)\|_p = \left(\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^p d\tau\right)^{\frac{1}{p}}$
- $\|e(t)\|_\infty = \sup_\tau (\max_i (e_i(\tau)))$ ↳ The supremum is the least upper bound

Singular Value Decomposition (SVD)

$$A = U\Sigma V^*$$

- $A \in \mathbb{C}^{l\times m}$, • $U \in \mathbb{C}^{l\times l}$, • $V \in \mathbb{C}^{m\times m}$, $\Sigma \in \mathbb{R}^{l\times m}$

① Compute $A^*A \in \mathbb{R}^{m\times m}$, symmetric!

② Find EW λ_i & EV v_i

③ Sort: $\lambda_1 > \lambda_2 > \dots > \lambda_m \geq 0$
$\quad v_1, v_2, \dots, v_m$

④ $V = [v_1 \ v_2 \cdots v_m] \in \mathbb{C}^{m\times m}$ (normalized?)

⑤ $\Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \sigma_2 & & & 0 \\ & & \ddots & & \\ & & & \sigma_p & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{l\times m}$, $\sigma_i = \sqrt{\lambda_i}$ ↳ Same Dim as A!

? $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p \geq 0$, $p = \min[l, m]$

⑥ $u_i = \frac{1}{\sigma_i} \cdot A \cdot v_i$, $i = 1, \dots, r$, $\sigma_i \neq 0$

⑦ Form orthonormal Basis of $\mathbb{C}^{l\times l}$:

If $r < l$: $\{u_1, u_2, \dots, u_r, \underbrace{u_{r+1}, \dots u_l}_{\text{Gram-Schmid}}\}$

Ex: $\hat{u}_3 = e_3 - (u_1 \cdot e_3)u_1 - (u_2 \cdot e_3)u_2$, $u_3 = \frac{\hat{u}_3}{\|\hat{u}_3\|}$

⑧ $U = [u_1 \ u_2 \dots u_l] \in \mathbb{C}^{l\times l}$

- $V^* = \begin{bmatrix} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{bmatrix}$ Input Directions (orthogonal?)
- $U = \begin{bmatrix} | & | & | & \cdots & | \end{bmatrix}$ Output Directions (orthogonal?)

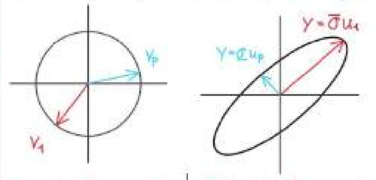
SVD Representation

$\sigma_i = \frac{\|Av_i\|}{\|v_i\|}$

- Input $\|u_i\| = 1$: $\quad$ • Output:

$v_1 = V_1$ (strong Input) $\quad$ $Y = \bar{\sigma}\, U_1$ (strong Output)

$v_p = V_p$ (weak Input) $\quad$ $Y = \underline{\sigma}\, U_p$ (weak Output)



Zero: $Y = Gv_{zero} = 0$ | Pole: $Y = Gv_{pole} = \infty$

Pole and Zero Direction from SVD

Pole: $P(s)|_{s=p_i} \cdot \delta_{p_i}^{in} = \infty \cdot \delta_{p_i}^{out}$

- $\underset{\epsilon\to 0}{P(p+\epsilon)} = \underbrace{(u_1\ u_2)}_{\text{output dir}} \begin{pmatrix} \sigma_1 \gg 0 & 0 \\ 0 & \sigma_2 \ll \sigma_1 \end{pmatrix} \underbrace{(v_1\ v_2)^*}_{\text{input dir}}$

Zero: $P(s)|_{s=z_i} \cdot \delta_{z_i}^{in} = 0 \cdot \delta_{z_i}^{out}$

- $P(z) = \underbrace{(u_1\ u_2)}_{\text{output dir}} \begin{pmatrix} \sigma_1 > 0 & 0 \\ 0 & \sigma_2 = 0 \end{pmatrix} \underbrace{(v_1\ v_2)^*}_{\text{input dir}}$

External Stability

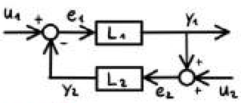
Closed Loop Transferfunction $T(s)$ has Poles in the left hand plane.

↳ BIBO: $\|y\| \leq k\|u\|$

Small gain theorem

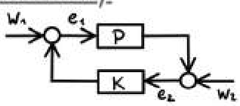
$\|L_1(s)\|_\infty = \gamma_1$

$\|L_2(s)\|_\infty = \gamma_2$



↳ stable if: $\gamma_1 \gamma_2 < 1$

Internal Stability

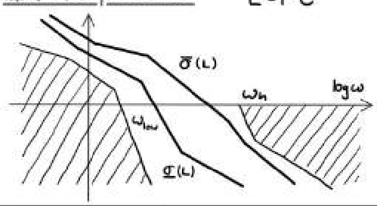


$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I-KP)^{-1} & (I-KP)^{-1}K \\ (I-PK)^{-1}P & (I-PK)^{-1} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

Internally stable if each TF is stable!

MIMO Performance

$L = P \cdot C$



Low Frequency

- Output Disturbance Attenuation: $d \to \eta$

  $\underline{\sigma}(C(j\omega)) \gg 1$

- Input Disturbance Attenuation: $d \to \nu$

  $\underline{\sigma}(C(j\omega)P(j\omega)) \gg 1$

- Good Tracking: $r \to e$

  $\underline{\sigma}(P(j\omega)C(j\omega)) \gg 1$

High Frequency

- Output Noise Rejection: $n \to \eta$

  $\bar{\sigma}(P(j\omega)C(j\omega)) \ll 1$ Robust Stability

- Input Noise Rejection: $n \to \nu$

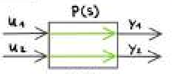
  $(\underline{\sigma}(P(j\omega)) \gg 1)$ Can't be controlled!

- Actuator Saturation:

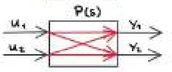
  $\bar{\sigma}(C(j\omega)) \ll M$

De-/Coupled Systems

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} P_{11} & 0 \\ 0 & P_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$



$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$$



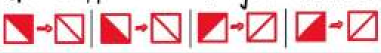
Relative Gain Array (RGA)

$RGA(P) = P \times (P^+)^T$, $P \in \mathbb{C}^{l\times m}$

$$\lambda_{ij} = \frac{\text{gain from } u_j \text{ to } y_i\text{, all other loops open}}{\text{gain from } u_j \text{ to } y_i\text{, all other loops closed}}$$

If $\lambda_{ij} = 1$: This channel is a SISO System.

If P is upper or lower triangular: $RGA(P) = \mathbb{I}$



Tall System ($P \in \mathbb{C}^{l\times m}$, $l > m$)

Left Moore-Penrose pseudo-inverse:

$P^+ = (P^*P)^{-1}P^*$, $Rank(P) = m$

Fat System ($P \in \mathbb{C}^{l\times m}$, $l < m$)

Right Moore-Penrose pseudo-inverse:

$P^+ = P^*(PP^*)^{-1}$, $Rank(P) = m$

General (2×2)-System

$$RGA(P) = \begin{bmatrix} \lambda_{11} & 1-\lambda_{11} \\ 1-\lambda_{11} & \lambda_{11} \end{bmatrix}, \quad \lambda_{11} = \frac{1}{1 - \frac{P_{12}P_{21}}{P_{11}P_{22}}}$$

RGA interpretation and pairing

Relevant Frequecies to evaluate RGA:

- steady state: $s = 0$
- crossover region: $s \in [j\omega_c, j\bar{\omega}_c]$

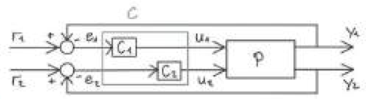
λ_{ij}	Interpretation	Pairing
1	No interaction from other channels	Yes
0	No effect from j-input to i-output	No
(0,1)	Closing loop increases channel gain	Best if ~1
>1	Closing loop decreases channel gain. The higher the more interaction	Best if ~1
<0	Closed & Open loop gain have opposite sign	No

Diagonal Dominance (RGA)

$RGA\text{-}Nr = \sum_{ij} |RGA(P) - \mathbb{I}|$

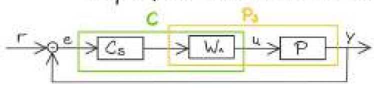
↳ low number → diagonally dominant system

Decentralized Control



Decoupeling Control

Intuition: use a "pre-compensator" to diagonalize the plant, then use decentralized control.



① $Y(s) = \underbrace{P(s)W_1(s)}_{P_d(s)}U(s)$

② $C(s) = W_1(s)C_s(s) = W_1(s)c(s)\mathbb{I}$

Find $W_1 = \begin{bmatrix} 1 & \delta_{12} \\ \delta_{21} & 1 \end{bmatrix}$:
- $W_1(s) = P^{-1}(s)$
- $W_1(0) = P^{-1}(0)$
- $W_1(j\omega_c) = P^{-1}(j\omega_c)$

Internal Model Control (IMC)

Q-Parametrization:

- $Q(s) = C(s)(\mathbb{I} + P(s)C(s))^{-1}$
- $C(s) = (\mathbb{I} - Q(s)P(s))^{-1}Q(s) = Q(s)(\mathbb{I} - P(s)Q(s))^{-1}$

Q-internal stability theorem:

- P(s) stable Plant of negative feedback sys. then sys. is internally stable if and only if Q(s) is stable.
- Easy to tune controller, as the sensitivity functions is linearly dependent on Q(s).
- For stable Q(s), stability is guaranteed even if C(s) is unstable.
- Non-minimum phase zeros in the Plant lead to unstable Q-parametrisation.

Linear Quadratic Regulator (LQR)

LQR minimizes J_{LQR}:

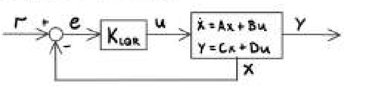
$$J_{LQR} = \int_0^\infty u(t)^T R u(t) + x(t)^T Q x(t)\, dt \quad \left(+2x(t)Nu(t)\right)$$

R & Q are diagonal, pos. definite matricies!

Q: cost of a non-zero state.

R: penalizes control effort (input).

Solution to LQR



$K_{LQR} = R^{-1}(N + PB)^T$ | $u_{LQR}(t) = -K_{LQR}x(t)$

Condition:

- System (A,B) is stabilizable!
- $(\tilde{A}, \tilde{Q}) = (A - BR^{-1}N^T, Q - NR^{-1}N^T)$ detectable!

Algebraic Riccati Equations (ARE)

Finding P: P is the real, symmetric, positive definite solution of the ARE

$$A^TP + PA - (N + PB)R^{-1}(N^T + B^TP) + Q = 0$$

$$\tilde{A}^*P + P\tilde{A} + P\tilde{R}P + \tilde{Q} = 0$$

$\tilde{A} = A - BR^{-1}N^T$

$\tilde{R} = -BR^{-1}B^T$ $\qquad P_{2\times 2} = \begin{bmatrix} \varphi_1 & \varphi_2 \\ \varphi_2 & \varphi_3 \end{bmatrix}$

$\tilde{Q} = Q - NR^{-1}N^T$

Hamiltonian

$$H = \begin{bmatrix} \tilde{A} & \tilde{R} \\ -\tilde{Q} & -\tilde{A}^* \end{bmatrix} \qquad \begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = \begin{bmatrix} V_{\lambda_1} \cdots V_{\lambda_n} \end{bmatrix}$$

$H \in \mathbb{C}^{2n\times 2n}$, $\quad \mathcal{X}_1, \mathcal{X}_2 \in \mathbb{C}^{n\times n}$

V_{λ_i} are the eigenvectors of H from the negative Eigenvalues. (only real λ_i)

$\Rightarrow P = \mathcal{X}_2 \mathcal{X}_1^{-1}$

Properties

P is symmetric, real, positive definite

$A - BK_{LQR} = H \rightarrow$ asymtotically stable

LQR Properties (N=0)

- infinite positive gain margin (in dB)
- negative gain margin of $20\log(0.5) = -6dB$
- 60° phase margin (both directions)
- robust stability to additive model uncert.

→ ∀ N: LQR controllers are inherently robust to model uncertainty!

Linear Quadratic Estimator (LQE)

The Kalman Filter (KF) is the optimal estimator for LTI-systems perturbed by zero mean, Gaussian, process and measurement noises.

→ solve exactly like LQR but with:

$B \to C^T$, $A \to A^T$, $K \to L^T$ (if N=0)

Condition:

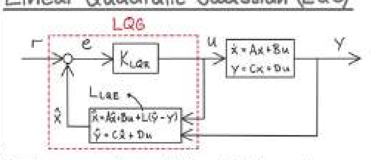
- System (A,C) is detectable!
- System (A,Q) is stabilizable!

Solution to LQE (if N=0)

$$AP + PA^T - PC^TR^{-1}CP + Q = 0$$

$\rightarrow L = PC^TR^{-1}$

Linear Quadratic Gaussian (LQG)



LQG: guarantees stability of CL-system.

- high level approach → macroscopic behaviour
- LQR & LQE provide solid rubustness guarantees → LQG does NOT!

Discrete LTI stability

$x[k+1] = r \cdot x[k]$

- $r \in (-\infty, -1] \to$ Div./alt.
- $r \in (-1, 0) \to$ Con./alt.
- $r \in (0, 1) \to$ Con./ap.
- $r \in [1, \infty) \to$ Div./ap.

$x[k+1] = A_D x[k] + \cdots$

- $\det(A_D - \lambda_i \mathbb{I}) = 0$

Con: $|\lambda_i| < 1$ | Ap: $\lambda_i \geq 0$

Div: $|\lambda_i| > 1$ | Alt: $\lambda_i < 0$

Continous A to Discret A_D

$S: [v_1\ v_2]$ eigenvector

- $e^A = S \cdot diag(e^{\lambda_1}\ e^{\lambda_2}) \cdot S^{-1}$
- $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{c-a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a-c} \\ 0 & 1 \end{bmatrix}$

Small gain theorem & Norm calculation

$\|L(j\omega)\|_\infty \rightarrow$ Biggest abs. value for $\forall\omega$

Discretization observation

In all cases: $s = 0 \to z = 0$: steady state is not affectet. (DC-Gain)

Euler forward: (c.t.) stable poles can map to unstable poles in (d.t.)

Euler backward: (c.t.) marginally stable poles can map to stable poles in (d.t.)

Tustin: (c.t.) stable poles map to (d.t.) stable poles

Matrix Norm

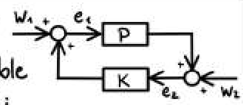
→ if: $\|A \cdot B\| \leq \|A\| \cdot \|B\|$

Decay of Error

$\dot{e} = (a - b - \ell c)e \rightarrow \int_{e(0)}^{e(t)} \frac{de}{e} = \int_0^t a - b - \ell c\, dt$

$\Rightarrow \ln\left(\frac{e(t)}{e(0)}\right) = (a - b - \ell c)t \qquad \frac{e(t)}{e(0)}$: decay of error

Closed Loop



Internally stable if and only if:

- $S_o(s) = (\mathbb{I} - P(s)K(s))$ is stable and
- there are no RHP cancellation in $P(s) \cdot K(s)$

If K(s) is minimum phase, we cannot conclude anything about internal stability!

Condional Number

- A measure of "directionality".

Duality

- Duality is a concept that allows to derive conditions on observability based on those for reachability.

Convert Dezibel

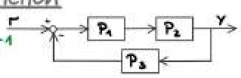
- $X\,dB = -20dB \cdot \log_{10}(y)$ $\quad$ - $Y = 10^{-\frac{X dB}{20 dB}}$

Singular Value & Input/Output

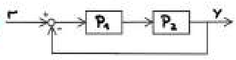
- $\underline{\sigma}(P(j\omega)) \leq \frac{\|y\|_2}{\|u\|_2} \leq \bar{\sigma}(P(j\omega))$

Transfer Function

$P_2P_1(\mathbb{I} + P_3P_2P_1)^{-1}$



$P_2P_1(\mathbb{I} + P_2P_1)^{-1}$



$P_1(\mathbb{I} + P_2P_1)^{-1}$